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Research Statement

Much of mathematics aims at representing a complex mathematical object by one which is more easily understood. This is the guiding light for my research. I am particularly interested in ways that quivers can be used to study much more complicated objects. My current work largely deals with tilting, cluster-tilting, and derived categories. In §1, I present the basic history and background leading to my thesis work, while in §2, I will briefly describe some of my results and some possible immediate extensions, lastly §3, I will briefly describe my current and future interests.

1. Background

1.1. **History.** In 2002, Fomin and Zelevinsky [19] introduced cluster algebras in the hopes of providing a new algebraic framework to study Lusztig's dual canonical basis. The original definition is elementary but the calculations can quickly become involved. To further study cluster algebras, cluster categories and cluster-tilted algebras were introduced in [8–10]. These categories and algebras allow us to study cluster algebras via the representation theory of quivers. Further, by [2,6] these algebras are closely related to the well studied class of tilted algebras. I am interested in exploring this connection for a particular class of cluster-tilted algebras which can be realized as triangulated surfaces (see [1,10,20]). This realization in terms of surfaces allows for a new combinatorial description of the representation theory of these algebras.

1.2. **Notation, Algebras, and Quivers.** Throughout we fix an algebraically closed field k. A quiver Q is the quadruple (Q_0, Q_1, s, t) of vertices, arrows, source and target functions on arrows. By a path we mean a directed sequence of arrows in Q. To any quiver we can define the *path algebra* kQ. The motivation for studying quivers in this context is that for any finite-dimensional algebra A, there is a bound quiver (Q, I) such that the representation theory of kQ/I is equivalent to the module theory of A for some ideal $I \subset kQ$, [4]. This equivalence allows for a description of the modules of A in terms of certain collections of vector spaces and linear maps.

1.3. **Surfaces and Triangulations.** Let S be a connected, oriented, unpunctured Riemann surface with boundary ∂S and let M be a non-empty finite subset of the boundary ∂S such that every boundary component contains at least one point of M. The elements of M are called *marked points.* We will refer to the pair (S, M) simply as an *unpunctured surface.* If M contains a point from the interior of S, this point is called a puncture.

We say that two curves in S *do not cross* if they do not intersect each other except at the endpoints at which they may coincide. We reserve the term *arc* for those non-self-intersecting curves in the interior of S with endpoints in M. A *generalized arc* may have self-intersections. Each generalized arc is considered up to homotopy inside the class of such curves. These arcs will encode information about modules of the algebra B_T which is defined below.

A *triangulation* is a maximal collection of non-crossing arcs, and we refer to the triple (S, M, T) as a *triangulated surface.* If $T = \{\tau_1, \tau_2, \ldots, \tau_n\}$ is a triangulation of an unpunctured surface (S, M), we define a quiver Q_T as in Figure 1. The vertices of Q_T are given by the edges τ_i and the arrows by the angles in each triangle. Note that the internal triangles in T correspond to oriented 3-cycles in Q_T.

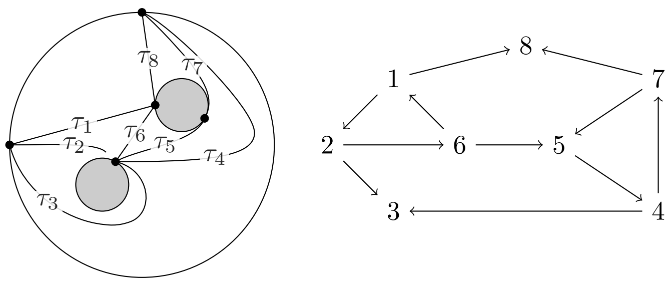

FIGURE 1. A triangulation and its quiver

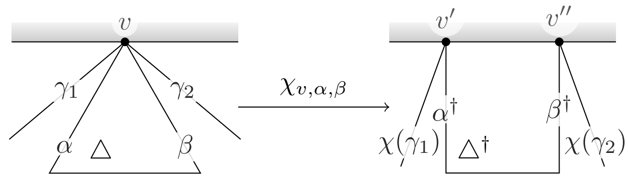

FIGURE 2. A local cut at v relative to α, β. The internal triangle $\triangle$ in T becomes a quadrilateral $\triangle^\dagger$ in $T^\dagger$.

Definition 1. Let (S, M, T) be a triangulated surface, as above. Define B_T to be the algebra which is defined as the quotient of the path algebra of the quiver Q_T by the two-sided ideal generated by the subpaths of length two in each oriented 3-cycle of Q_T.

In [17], the authors associate a cluster algebra $\mathcal{A}(Q_T)$ to this quiver; the cluster algebras obtained in this way are called cluster algebras from (unpunctured) surfaces ([16–18, 22, 23]), and the corresponding cluster categories in [7, 10]. Each triangulation of a surface corresponds to a cluster-tilted algebra or a 2-Calabi-Yau tilted algebra. When S is a disc, Q_T corresponds to a cluster-tilted algebra of Dynkin type A. When S is an annulus, we get the cluster-tilted algebras of affine Dynkin type $\tilde{A}$. For all other surfaces, we obtain 2-Calabi-Yau tilted algebras. I am particularly interested in the last two cases because of the following theorem.

Theorem 2 ([6]). *An algebra B with global dimension at most 2 is iterated tilted of Dynkin type Q if and only if it is the quotient of a cluster-tilted algebra of type Q by an admissible cut.*

In the situation of interest, algebras coming from triangulations of surfaces, an admissible cut is given by removing an arrow from each oriented three cycle. I have shown how to translate this theorem into a construction on the surface.

2. My Results

2.1. **Defining Surface Algebras.** Throughout this section, we assume that if S is a disc, then M has at least 5 marked points. To translate Theorem 2 into the setting of triangulated surfaces, we define the admissible cut of a triangulation. To that purpose, we introduce *local cuts* of a surface via Figure 2. This definition can be made rigorous, but the figure gives the morally correct idea.

If the algebra B in Theorem 2 corresponds to a triangulation of a surface, its admissible cuts correspond to a sequence of local cuts at *each* internal triangle of the triangulation. See Figure 3.

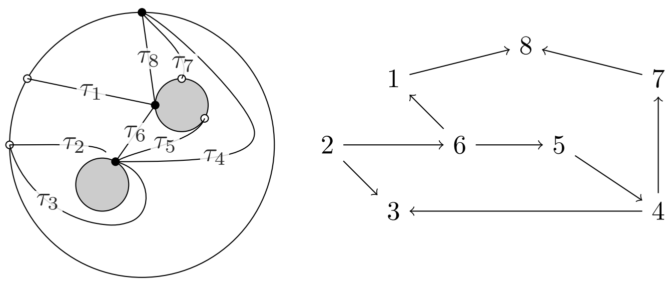

FIGURE 3. An admissible cut in the surface and its quiver. The white marked points indicate the local cuts. There are two local cuts in this figure, one between τ_1 and τ_2, the other between τ_7 and τ_5.

Definition 3. A *surface algebra of type* (S, M) is a bound quiver algebra $B_{T^\dagger} = kQ_{T^\dagger}/I_{T^\dagger}$ where $(S, M^\dagger, T^\dagger)$ is the partial triangulation given by a sequence of local cuts of a triangulated unpunctured surface (S, M, T). The ideal $I^\dagger$ is generated by the paths of length two defined by the quadrilaterals of the partial triangulation.

Theorem 4 ([12])**.** *If (S, M, T) is a triangulated disc and $(S, M^\dagger, T^\dagger)$ an admissible cut, then $B_{T^\dagger}$ is an iterated tilted algebra of Dynkin type A.*

The surface algebras of the disc are well-known, but we we see in Theorem 6 that this construction provides an interesting approach to study the representation theory of these algebras. Surface algebras from surfaces other than the disc seem to be new as they do not fit into any known classification of algebras. However, as desired, these algebras parallel the relationship between tilted an cluster-tilted algebras seen in [2,6].

Theorem 5 ([12])**.** *If $(S, M^\dagger, T^\dagger)$ is an admissible cut of (S, M, T) then*
(a) *$Q_{T^\dagger}$ is an admissible cut of Q_T.*
(b) *The tensor algebra of $B_{T^\dagger}$ with respect to the $B_{T^\dagger}$-bimodule $\mathrm{Ext}^2_{B_{T^\dagger}}(DB_{T^\dagger}, B_{T^\dagger})$ is isomorphic to the algebra B_T.*

The partially triangulated surfaces also provide a convenient graphical tool for presenting the modules of these algebras.

Theorem 6 ([12])**.** *The surface algebras are gentle. Moreover, the string modules of $B_{T^\dagger}$ correspond to the generalized arcs of $(S, M^\dagger, T^\dagger)$.*

3. FUTURE

My primary focus has been mostly restricted to the genus 0 case; these are surfaces that can be thought of as a deformed sphere that have had at least one disc removed from it. This of course leaves most questions involving higher genus open. In particular I have shown [11] that derived equivalence of surface algebras of genus 0 can be determined entirely by a combinatorial method that depends only on the configuration of cuts in the surface. This is partially achieved by calculating an invariant defined by Avella-Alaminos and Geiss in [5] as a function of the number of marked points and the number of local cuts on a boundary component. Using the work of [3] on graded equivalence, we can then determine derived equivalences using distribution of the local cuts throughout the boundary components. Unfortunately, this method cannot be extended to higher genus. In fact, a simple counter-example can be found by considering the torus with a single disc removed. There is something significantly more complicated about the higher genus case.

Similarly, I have so far restricted my study to surfaces without punctures. When cutting in the boundary, we are inserting a new boundary segment. It is not clear what the corresponding action should be when cutting at a point in the interior of the surface. These algebras are of interest because the cluster-tilted algebras of Dynkin type D and affine $\tilde{D}$ are given by triangulations of once and twice punctured discs, respectively. I currently believe that the correct analogue for a local cut at a puncture m is to introduce an open boundary segment in the interior of S and incident to m, which we call an incision at m.

Conjecture 7. *Taking the incision of a punctured surface at a puncture provides the correct combinatorial model to describe the iterated-tilted algebras of type D.*

The punctured case also presents a difficulty in that the invariant of Avella-Alaminos and Geiss [5] need not be defined for these algebras. It is not clear if it can be easily generalized.

Beyond extensions to the punctured case, there are also the m-angulated surfaces used in describing m-cluster categories and the corresponding m-cluster-tilted algebras. When the surface is the disc, then much of the work in my thesis extends quite readily.

Conjecture 8. *The notion of the local cuts and the calculation of the invariant of Avella-Alaminos and Geiss can be extended for m-angulations of the disc.*

However, the theory of m-angulated surfaces with more than one boundary component is not as well established as in the triangulated case.

Beyond working with surfaces I am also interested in exploring how/if quivers can be used to detect other invariants, such as the dimension or the radius of the module category defined in [13, 14]. I have also begun exploring the uses of quivers in mathematical physics. I am particularly interested in [15, 21].

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